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Inverse spectral problems for left-definite Sturm–Liouville equations with indefinite weight

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Abstract

Inverse Sturm–Liouville problems with indefinite weight are considered. Theorems analogous to those of both Hochstadt and Gelfand–Levitan are proved. As a sample result, under suitable symmetry conditions, we show that half of one spectrum suffices to determine the potential function uniquely. © 2002 Elsevier Science (USA). All rights reserved.

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1. Introduction

Sturm–Liouville theory can be regarded as splitting into two branches, direct and inverse. The direct, or 'forward,' theory starts with the coefficients $p > 0$, q and r in the equation

$$-(py')' + qy = \lambda ry \quad \text{on } [a, b], \tag{1.1}$$

where we shall take $a < 0 < b$, and the boundary conditions

$$y(a)\cos\alpha + (py')(a)\sin\alpha = 0, \tag{1.2}$$

$$y(b)\cos\beta + (py')(b)\sin\beta = 0, \tag{1.3}$$

and then derives the properties of the eigenvalues $\lambda = \lambda_n$ (e.g., that they form a countable sequence λ_n, accumulating only at $+\infty$ and $-\infty$) and the corresponding eigenfunctions $y = y_n$ (e.g., that they are unique up to scaling). The literature on such problems is vast for the case when r is of one sign, but even when r is indefinite, there are many investigations, particularly for cases with one turning point (i.e., where r changes sign). For applications we refer to [2,5, 12] and references therein; qualitative theory of eigenvalues and eigenfunctions is discussed in [1,15]. For completeness and expansion (including half range) theory, see [2,8,9,17,19].

The inverse theory starts with certain properties of the eigenvalues and eigenfunctions and deduces properties of the coefficients in (1.1)–(1.3). This theory also splits into different directions, e.g., existence and reconstruction of the coefficients given specific properties of λ_n and y_n, and we refer to the survey paper [24] and to the books [23] and [25] for various aspects of this field. Our work concerns 'uniqueness' for the inverse problem. Early papers on this question specified enough spectral information to determine q uniquely, for given p and r (usually both identically 1) and boundary conditions. The seminal work of Borg [6], where the eigenvalues λ_n are specified for (1.1)–(1.3), together with a second set of eigenvalues λ'_n for a similar problem with β replaced by β', is of this type. We remark that uniqueness results provide the necessary underpinning for investigations of the reconstruction question (see above) to be meaningful.

Although Borg's results were not the first in inverse Sturm–Liouville theory, [6] can be viewed as the foundation stone for the subject. We refer to the somewhat complementary surveys in [23] and [24] for further activity in this area, but two papers are particularly relevant to our study. One, by Gelfand and Levitan [13], specifies the eigenvalues λ_n for just one set of boundary conditions, together with the corresponding 'norming constants' $\|y_n\|^2/|y_n(a)|$, for the case of $p = r = 1$. The other, by Hochstadt [14], refines the approach of Levinson [22] to show (given certain restrictions on p and r which must be positive) precisely how much freedom q has if the λ_n and all but finitely many of the λ'_n (see above) are specified. This modified 'uniqueness' result seems to be the most general of its type, and we shall focus on it together with the above result of [13], for a more general weight function r which can change sign.

Inverse problems for indefinite weight Sturm–Liouville problems seem comparatively undeveloped, but have important applications including the case where the weight function r is discontinuous at a turning point. As an example, one could consider the recovery of the refractive index from wave propagation in an inhomogeneous isotropic medium consisting of two strata separated by the plane $x = 0$, where one stratum admits absorption and the other does not. (For physical aspects of wave propagation through inhomogeneous media, see [16, Chapter 6].) Thus we shall allow r to be discontinuous (in contrast to [12] where $r \in C^2$ was assumed).

In Sections 2 and 3 we shall set up our assumptions precisely and develop the foundations of our analysis. We recast (1.1)–(1.3) in the form $AY = \lambda Y$ for a self-adjoint operator A in a suitable function space; cf. [10,18]. Section 4 is devoted to the study of the transformation operator T which maps solutions of (1.1) obeying the boundary conditions (1.2) into solutions of

$$-(py')' + \tilde{q}y = \lambda ry \quad \text{on } [a, b] \tag{1.4}$$

also obeying (1.2). In previous work (see, e.g., [14]), T has acted between L^2 spaces, but here it acts on a subset of L^2 which is a Hilbert space when considered with the Dirichlet norm. Additional care is needed to cope with multiplication by r not being a continuously invertible map.

In Section 5 we give our main results. We extend Hochstadt's results from [14] to indefinite weight and combine our methods with the approach of [14] to obtain the uniqueness result associated with the names of Gelfand and Levitan (but actually proved in [20]). We also consider the case where q is symmetric and r and the boundary conditions are antisymmetric. Then just half of one spectrum suffices to determine q, in contrast with the results cited earlier requiring two complete spectra. The methods used in this work rely on understanding the nature of the Green's function of (1.1)–(1.3) in the complex λ-plane. We also lean heavily on forward asymptotics for the eigenvalues and various constructions based on the eigenfunctions in cases when r has one sign. These are collected in Appendix A.

2. Preliminaries

Let p be a C^1 positive function with absolutely continuous derivative on $[a, b]$, where $a < 0 < b$. Let r be a C^1 positive function with absolutely continuous derivative on $(0, b]$ and negative function with absolutely continuous derivative on $[a, 0)$. In addition assume that either $-\infty < r(0^-) < 0 < r(0^+) < \infty$ or that r is differentiable at 0, having $r'(0) > 0$. Let $q, \tilde{q}$ be real valued L^1 functions on $[a, b]$.

Let

$$ly = -(py')' + qy, \qquad \tilde{l}y = -(py')' + \tilde{q}y$$

and

$$Ry = ry.$$

Define $(A; \alpha; \beta)$ to be the eigenvalue problem

$$Ay = \lambda Ry, \quad 0 \neq y \in \mathcal{D}(A),$$

where the operator A is given by $Ay = ly$ with domain

$$\mathcal{D}(A) = \big\{y\colon\ y, py' \in AC[a,b],\ ly \in L^2[a,b], \\ y(a)\cos\alpha + py'(a)\sin\alpha = 0, \\ y(b)\cos\beta + py'(b)\sin\beta = 0\big\}$$

with $\alpha, \beta \in [0,\pi)$. Define $(\tilde{A}; \alpha; \beta)$, $\tilde{A}$ and $\mathcal{D}(\tilde{A})$ in an analogous manner with l replaced by $\tilde{l}$. Without loss of generality we assume that $p = 1$.

We make the additional assumption that A is a positive operator on $\mathcal{D}(A)$, which is equivalent to (1.1)–(1.3) being left-definite (conditions which ensure the positivity of A are given in [4]; these usually impose additional constraints on α and β). This allows us to take

$$\langle u, v\rangle = \int_a^b \bar{v} lu\, dt = \int_a^b u \overline{lv}\, dt \tag{2.1}$$

as the inner product on $\mathcal{D}(A)$. Let $|||u||| = \sqrt{\langle u, u\rangle}$, and denote by D the completion of $\mathcal{D}(A)$ with respect to $|||\cdot|||$. Then D is a Hilbert space which is continuously imbedded in $L^2(a,b)$.

For λ not in the spectrum of (1.1)–(1.3) let

$$S_\lambda = (A - \lambda R)^{-1} R|_D.$$

Define $\tilde{S}_\lambda$ analogously, replacing A by $\tilde{A}$.

Lemma 2.1. *The operator S_0 is self-adjoint on D, and for λ not in the spectrum of* (1.1)–(1.3), *S_λ is a compact operator on D.*

Proof. To see that S_λ is compact on D we reason in a manner similar to Lemma 2.2 of [3]. Let $x_n \rightharpoonup_D 0$; then $x_n \rightharpoonup_{L^2} 0$ (as A^{-1} is defined on all of L^2 since A is positive) and as r is a bounded function $Rx_n \rightharpoonup_{L^2} 0$. Then Rx_n is a bounded sequence in L^2 but $(A - \lambda R)^{-1}$ is compact on L^2; so $a_n := (A - \lambda R)^{-1} Rx_n \to_{L^2} 0$ and hence $Aa_n = Rx_n + \lambda Ra_n$ which is bounded in L^2. Thus

$$|||S_\lambda x_n|||^2 = (Rx_n + \lambda Ra_n, a_n) \to 0,$$

proving that S_λ is compact on D.

That S_0 is self-adjoint is easily verified. $\square$

The eigenvalue problem

$$S_0 y = \mu y, \quad y \in D, \tag{2.2}$$

is equivalent to (1.1)–(1.3) with $\lambda = 1/\mu$. Thus S_0 has a real spectrum and all points of the spectrum, with the possible exception of 0, are eigenvalues. From the structure of l and the boundary conditions it follows that each eigenvalue is simple. The eigenfunctions can thus be chosen so as to form an orthonormal basis for D.

Let w be the solution of $lw = \lambda r w$ obeying the initial conditions

$$w(a) = \sin\alpha, \qquad w'(a) = -\cos\alpha, \tag{2.3}$$

and v be the solution of $lv = \lambda r v$ obeying the terminal conditions

$$v(b) = \sin\beta, \qquad v'(b) = -\cos\beta. \tag{2.4}$$

Define $\tilde{w}$ and $\tilde{v}$ as above but with l replaced by $\tilde{l}$. Let

$$\psi(\lambda) = \cos\beta\, w(b;\lambda) + \sin\beta\, w'(b;\lambda),$$

and define $\tilde{\psi}(\lambda)$ in an analogous manner.

Theorem 2.2. *The zeros of the entire function ψ are simple and coincide with the eigenvalues of $(A;\alpha;\beta)$, which are real and have multiplicity* 1.

Proof. The spectrum of $(A;\alpha;\beta)$ is entirely determined by the spectrum of S_0. Hence we obtain that the spectrum of $(A;\alpha;\beta)$ consists entirely of real simple eigenvalues. That ψ is entire follows from [7], and that the zeros of ψ are precisely the eigenvalues of $(A;\alpha;\beta)$ is obvious. It remains only to prove that the zeros are simple.

Suppose λ to be a nonsimple zero of ψ. Then $\psi(\lambda) = 0 = \psi'(\lambda)$, which gives

$$0 = w(b)\cos\beta + w'(b)\sin\beta,$$
$$0 = w_\lambda(b)\cos\beta + {w_\lambda}'(b)\sin\beta,$$

where the subscript λ denotes partial differentiation with respect to λ. Combining the above equations we obtain

$$0 = [w' w_\lambda - w w'_\lambda](b). \tag{2.5}$$

It can also be verified that

$$0 = [w' w_\lambda - w w'_\lambda](a). \tag{2.6}$$

As $lw = \lambda r w$ for all λ,

$$-w'' + qw = \lambda r w$$

and

$$-w''_\lambda + q w_\lambda = r w + \lambda r w_\lambda.$$

Combining the above two equations, we obtain

$$w^2 r = w'' w_\lambda - w w_\lambda''.$$

Integration and use of (2.5), (2.6) give $0 = \int_a^b r w^2\, dt$. But as $\lambda \in \mathbb{R}$, $w(x) \in \mathbb{R}$ and so

$$0 \neq |||w|||^2 = \langle w, w \rangle = \int_a^b w l w\, dt = \lambda \int_a^b r w^2\, dt = 0. \qquad \square$$

Let

$$\mathbb{Z}^0 = \mathbb{Z} \backslash \{0\}.$$

Since the eigenvalues of $(A; \alpha; \beta)$ are real, simple and have no finite accumulation points (Theorem 2.2), and form a bi-infinite sequence [1], we may denote them by $\cdots < \lambda_{-2} < \lambda_{-1} < 0 < \lambda_1 < \lambda_2 < \cdots$. The function $w_n(x) = w(x; \lambda_n)$ is then an eigenfunction of $(A; \alpha; \beta)$ with eigenvalue λ_n for $n \in \mathbb{Z}^0$. For brevity, write $v_n(x) = v(x; \lambda_n)$. As w_n and v_n are linearly dependent for each $n \in \mathbb{Z}^0$, let k_n be such that

$$k_n w_n = v_n. \tag{2.7}$$

Define $\tilde{w}_n$, $\tilde{v}_n$ and $\tilde{k}_n$ in a similar manner.

Let γ be such that $\sin\gamma \neq \sin\beta$ and

$$\phi(\lambda) = w(b; \lambda) \cos\gamma + w'(b; \lambda) \sin\gamma. \tag{2.8}$$

Replacing ψ by ϕ in Theorem 2.2 we see that the zeros of ϕ are real and simple and are precisely the eigenvalues of $(A; \alpha; \gamma)$. Define $\tilde{\phi}$ in an analogous manner.

Lemma 2.3. *If $(A; \alpha; \gamma)$ and $(\tilde{A}; \alpha; \gamma)$ have the same eigenvalues then $\phi = \tilde{\phi}$.*

Proof. From [7] it follows that ϕ is an entire function, while from (A.9), (A.10) and (A.24), (A.25) with β replaced by γ we see that ϕ and $\tilde{\phi}$ are functions of order $1/2$ and, consequently, using Hadamard's theorem [21], are determined up to a multiplicative constant by their zeros. Hence there is a constant k such that $k = \phi(\lambda)/\tilde{\phi}(\lambda)$. From (A.9), (A.10) and (A.24), (A.25) we obtain $k = 1 + O(\lambda^{-1/2})$ for $\lambda \in i\mathbb{R}$. Hence $k = 1$ and $\phi = \tilde{\phi}$. $\square$

Let $\Lambda_0 \subseteq \mathbb{Z}^0$ be a finite set and $\Lambda = \mathbb{Z}^0 \backslash \Lambda_0$.

Theorem 2.4. *If $(A; \alpha; \gamma)$ and $(\tilde{A}; \alpha; \gamma)$ have the same eigenvalues and, in addition, $\lambda_n = \tilde{\lambda}_n$ for all $n \in \Lambda$, where λ_n and $\tilde{\lambda}_n$ are the eigenvalues of $(A; \alpha; \beta)$ and $(\tilde{A}; \alpha; \beta)$, respectively, then $k_n = \tilde{k}_n$ for all $n \in \Lambda$.*

Proof. From Theorem 2.2 and the definition of ϕ it follows that

$$\cos\beta\, w_n(b) + \sin\beta\, w_n'(b) = 0,$$
$$\cos\gamma\, w_n(b) + \sin\gamma\, w_n'(b) = \phi(\lambda_n).$$

The above linear system has a unique solution:

$$w_n(b) = -\frac{\sin\beta}{\sin(\gamma-\beta)}\phi(\lambda_n),$$
$$w_n'(b) = \frac{\cos\beta}{\sin(\gamma-\beta)}\phi(\lambda_n),$$

and similarly for $\tilde{w}_n(b)$ and $\tilde{w}_n'(b)$.

But $\lambda_n = \tilde{\lambda}_n$ for all $n \in \Lambda$ and from Lemma 2.3, $\phi = \tilde{\phi}$; so for $n \in \Lambda$ we have

$$w_n(b) = \tilde{w}_n(b)$$

and

$$w_n'(b) = \tilde{w}_n'(b).$$

From the definition of k_n, $\tilde{k}_n$, w_n and $\tilde{w}_n$ it follows that for all $n \in \Lambda$

$$k_n w_n(b) = \sin\beta = \tilde{k}_n \tilde{w}_n(b),$$
$$k_n w_n'(b) = -\cos\beta = \tilde{k}_n \tilde{w}_n'(b).$$

Since not both $w_n(b)$ and $w_n'(b)$ are zero, the result follows. □

3. Uniformly convergent expansions

This section is devoted to Theorem 3.1, which provides residue expansions for the kernel G/ψ of $(\lambda R - A)^{-1}$ and for the kernel K/ψ of the transformation operator T, to be defined in Section 4. These series expansions play a crucial role in the proofs of our subsequent results.

The proof of Theorem 3.1 uses a method of [26] and relies on the integration of holomorphic functions around contours in the complex plane. Bounds for these integrals involve functions with multiple branches. To avoid possible confusion in the interpretation of such functions we adopt the convention that by $[e^{i\theta}]^s$, for $\theta \in (-\pi, \pi]$ and $s > 0$, we mean $e^{is\theta}$.

Theorem 3.1. *Let*

$$G(x;y;\lambda) = \begin{cases} v(x;\lambda)w(y;\lambda), & a \leqslant y < x \leqslant b, \\ w(x;\lambda)v(y;\lambda), & a \leqslant x < y \leqslant b, \end{cases} \tag{3.1}$$

$$K(x;y;\lambda) = \begin{cases} \tilde{v}(x;\lambda)w(y;\lambda), & a \leqslant y < x \leqslant b, \\ \tilde{w}(x;\lambda)v(y;\lambda), & a \leqslant x < y \leqslant b. \end{cases} \tag{3.2}$$

Then for $\lambda \neq \lambda_n$, $n \in \mathbb{Z}^0$,

$$\frac{G(x;y;\lambda)}{\psi(\lambda)} = \sum_{j\in\mathbb{Z}^0} \frac{k_j w_j(x) w_j(y)}{(\lambda-\lambda_j)\psi'(\lambda_j)}, \tag{3.3}$$

$$\frac{K(x;y;\lambda)}{\psi(\lambda)} = \sum_{j\in\mathbb{Z}^0} \frac{K(x;y;\lambda_j)}{(\lambda-\lambda_j)\psi'(\lambda_j)}, \tag{3.4}$$

where the above summations converge uniformly for $(x, y) \in [a, b]^2$. *In addition*

$$\frac{[\tilde{v}w-\tilde{w}v](x;\lambda)}{\psi(\lambda)} = \sum_{j\in\mathbb{Z}^0} \frac{[\tilde{v}-k_j\tilde{w}](x;\lambda_j)w_j(x)}{(\lambda-\lambda_j)\psi'(\lambda_j)}, \tag{3.5}$$

where the above summation converges uniformly for $x \in [a, b]$.

Proof. *Case* I: $r(0^-) < 0 < r(0^+)$.

For $n \in \mathbb{N}$ let

$$\Gamma_n = \Gamma_n^{-1} \cup \Gamma_n^0 \cup \Gamma_n^1, \tag{3.6}$$

where

$$\Gamma_n^0 = \bigl\{\bigl(\Theta_1(n)+i\zeta\bigr)^2\colon\ \zeta \in \bigl[-\Theta_2(n), \Theta_2(n)\bigr]\bigr\}, \tag{3.7}$$

$$\Gamma_n^{-1} = \bigl\{\bigl(\zeta - i\Theta_2(n)\bigr)^2\colon\ \zeta \in \bigl[0, \Theta_1(n)\bigr]\bigr\}, \tag{3.8}$$

$$\Gamma_n^1 = \bigl\{\bigl(-\zeta + i\Theta_2(n)\bigr)^2\colon\ \zeta \in \bigl[-\Theta_1(n), 0\bigr]\bigr\}, \tag{3.9}$$

and

$$\Theta_1(n) = \begin{cases} \frac{\kappa+(n-1/2)\pi}{\int_0^b \sqrt{r}}, & \sin\beta \neq 0, \\ \frac{\kappa+n\pi}{\int_0^b \sqrt{r}}, & \sin\beta = 0, \end{cases} \tag{3.10}$$

$$\Theta_2(n) = \begin{cases} \frac{-\kappa+n\pi}{\int_a^0 \sqrt{-r}}, & \sin\alpha \neq 0, \\ \frac{-\kappa+(1/2+n)\pi}{\int_a^0 \sqrt{-r}}, & \sin\alpha = 0, \end{cases} \tag{3.11}$$

where $\kappa = \arctan\sqrt{|r(0^-)/r(0^+)|} \in (0, \pi/2)$ (see also Theorem A.1). For brevity the remainder of the proof will be for the case of $\sin\alpha \neq 0 \neq \sin\beta$; the proofs for the other three cases are similar in nature and can easily be constructed using the above definitions of Θ_1 and Θ_2.

Using (A.9), (A.10) and the notation from (3.7)–(3.9) we have for $\lambda \in \Gamma_n^0$, n large,

$$\bigl|\psi(\lambda)\bigr| \geqslant c_0\sqrt{|\lambda|}\exp\left(\Theta_1(n)\int_a^0 \sqrt{|r|} + \zeta\int_0^b \sqrt{r}\right) \tag{3.12}$$

for some constant $c_0 > 0$, and for $\lambda \in \Gamma_n^1 \cup \Gamma_n^{-1}$

$$\left|\psi(\lambda)\right| \geqslant c_1\sqrt{|\lambda|}\exp\left(\Theta_2(n)\int_0^b \sqrt{r} + |\zeta|\int_a^0 \sqrt{|r|}\right)$$

for some $c_1 > 0$.

Using (A.1), (A.8) and the notation from (3.7)–(3.9) again, but this time in connection with (3.1), (3.2) and $[\tilde{v}w - \tilde{w}v](x;\lambda)$, we obtain for $\lambda \in \Gamma_n^0$

$$G(x;y;\lambda) = \begin{cases} O\left(\exp\left[\Theta_1(n)\int_a^x \sqrt{|r|} + \zeta\int_y^b \sqrt{r}\,\right]\right), & x < 0 < y, \\ O\left(\exp\left[\Theta_1(n)\left(\int_a^x \sqrt{|r|} + \int_y^0 \sqrt{|r|}\right) + \zeta\int_0^b \sqrt{r}\,\right]\right), & x < y < 0, \\ O\left(\exp\left[\Theta_1(n)\int_a^0 \sqrt{|r|} + \zeta\left(\int_y^b \sqrt{r} + \int_0^x \sqrt{r}\,\right)\right]\right), & 0 < x < y\,, \end{cases}$$

and for $\lambda \in \Gamma_n^1 \cup \Gamma_n^{-1}$

$$G(x;y;\lambda) = \begin{cases} O\left(\exp\left[|\zeta|\int_a^x \sqrt{|r|} + \Theta_2(n)\int_y^b \sqrt{r}\,\right]\right), & x < 0 < y, \\ O\left(\exp\left[|\zeta|\left(\int_a^x \sqrt{|r|} + \int_y^0 \sqrt{|r|}\right) + \Theta_2(n)\int_0^b \sqrt{r}\,\right]\right), & x < y < 0, \\ O\left(\exp\left[|\zeta|\int_a^0 \sqrt{|r|} + \Theta_2(n)\left(\int_y^b \sqrt{r} + \int_0^x \sqrt{r}\,\right)\right]\right), & 0 < x < y, \end{cases}$$

and similarly for $K(x;y;\lambda)$.

Finally, we obtain for $\lambda \in \Gamma_n^0$

$$[\tilde{v}w - \tilde{w}v](x;\lambda) = O\left(\exp\left[\Theta_1(n)\int_a^0 \sqrt{|r|} + \zeta\int_0^b \sqrt{r}\right]\right),$$

and for $\lambda \in \Gamma_n^1 \cup \Gamma_n^{-1}$

$$[\tilde{v}w - \tilde{w}v](x;\lambda) = O\left(\exp\left[|\zeta|\int_a^0 \sqrt{|r|} + \Theta_2(n)\int_0^b \sqrt{r}\right]\right).$$

It is to be noted that all the constants implied by the symbol $O(\cdot)$, in the above, are independent of x and y. Γ_n is a closed curve in the λ-plane, and for large n, Γ_n encloses $\lambda_{-n}, \dots, \lambda_{-1}, \lambda_1, \dots, \lambda_n$ and excludes λ_j, $|j| > n$. Let $\mu \in \mathbb{C}\setminus\{\lambda_j\colon\ j \in \mathbb{Z}^0\}$ and $n \in \mathbb{N}$ be so large that Γ_n encloses μ. Then from the residue theorem and the above bounds,

$$\frac{G(x;y;\mu)}{\psi(\mu)} + \sum_{0<|j|\leqslant n} \frac{G(x;y;\lambda_j)}{(\lambda_j - \mu)\psi'(\lambda_j)} = \frac{1}{2\pi i}\int_{\Gamma_n} \frac{G(x;y;\lambda)}{(\lambda - \mu)\psi(\lambda)}\,d\lambda = O\left(\frac{1}{n}\right),$$

and similarly for $K(x;y;\mu)/\psi(\mu)$. Finally,

$$\frac{[\tilde{v}w-\tilde{w}v](x;\mu)}{\psi(\mu)}+\sum_{0<|j|\leqslant n}\frac{[\tilde{v}w-\tilde{w}v](x;\lambda_j)}{(\lambda_j-\mu)\psi'(\lambda_j)}$$
$$=\frac{1}{2\pi i}\int\limits_{\Gamma_n}\frac{[\tilde{v}w-\tilde{w}v](x;\lambda)}{(\lambda-\mu)\psi(\lambda)}\,d\lambda=O\left(\frac{1}{n}\right).$$

By letting $n\to\infty$ in the above and noting that $k_jw_j(x)=v_j(x)$ for all $j\in\mathbb{Z}^0$, we obtain the required summations. Noting again that the constants implied by $O(\cdot)$ are independent of x and y, we obtain the stated uniform convergence. This completes the proof of the theorem for the case of $r(0^-)<0<r(0^+)$.

Case II: $r(x)=xr_1(x)$, where r_1 is continuous and positive on $[a,b]$.

Let

$$\Theta_1(n)=\begin{cases}\frac{(1/4+n)\pi}{\int_0^b\sqrt{r}}, & \sin\beta\neq 0,\\ \frac{(n-1/4)\pi}{\int_0^b\sqrt{r}}, & \sin\beta=0,\end{cases}\tag{3.13}$$

$$\Theta_2(n)=\begin{cases}\frac{-(1/4+n)\pi}{\int_a^0\sqrt{-r}}, & \sin\alpha\neq 0,\\ \frac{-(n-1/4)\pi}{\int_a^0\sqrt{-r}}, & \sin\alpha=0.\end{cases}\tag{3.14}$$

It can be shown that for $\lambda\in\Gamma_n^0$

$$\left|\psi(\lambda)\right|\geqslant\begin{cases}c_0|\lambda|^{1/2}\exp\bigl(\Theta_1(n)\int_a^0\sqrt{|r|}+\zeta\int_0^b\sqrt{r}\bigr), & \sin\alpha\neq 0\neq\sin\beta,\\ c_0\exp\bigl(\Theta_1(n)\int_a^0\sqrt{|r|}+\zeta\int_0^b\sqrt{r}\bigr), & \sin\alpha=0\neq\sin\beta,\\ c_0\exp\bigl(\Theta_1(n)\int_a^0\sqrt{|r|}+\zeta\int_0^b\sqrt{r}\bigr), & \sin\alpha\neq 0=\sin\beta,\\ c_0|\lambda|^{-1/2}\exp\bigl(\Theta_1(n)\int_a^0\sqrt{|r|}+\zeta\int_0^b\sqrt{r}\bigr), & \sin\alpha=0=\sin\beta,\end{cases}$$

for some positive constant c_0, and for $\lambda\in\Gamma_n^1\cup\Gamma_n^{-1}$

$$\left|\psi(\lambda)\right|\geqslant\begin{cases}c_1|\lambda|^{1/2}\exp\bigl(\Theta_2(n)\int_0^b\sqrt{r}+|\zeta|\int_a^0\sqrt{|r|}\bigr), & \sin\alpha\neq 0\neq\sin\beta,\\ c_1\exp\bigl(\Theta_2(n)\int_0^b\sqrt{r}+|\zeta|\int_a^0\sqrt{|r|}\bigr), & \sin\alpha=0\neq\sin\beta,\\ c_1\exp\bigl(\Theta_2(n)\int_0^b\sqrt{r}+|\zeta|\int_a^0\sqrt{|r|}\bigr), & \sin\alpha\neq 0=\sin\beta,\\ c_1|\lambda|^{-1/2}\exp\bigl(\Theta_2(n)\int_0^b\sqrt{r}+|\zeta|\int_a^0\sqrt{|r|}\bigr), & \sin\alpha=0=\sin\beta,\end{cases}$$

for some positive constant c_1. For the analysis of G, K and $\tilde{v}w-\tilde{w}v$ we present bounds for $wv(x;\lambda)$ when $a\leqslant x\leqslant 0$; the case of $0\leqslant x\leqslant b$ is similar but with the roles of v and w interchanged. By similar reasoning it can be shown that bounds derived apply to $w(x;\lambda)v(y;\lambda)$, $\tilde{w}(x;\lambda)v(y;\lambda)$ and $\tilde{v}(y;\lambda)w(x;\lambda)$ for all $a\leqslant x\leqslant y\leqslant b$.

The approach is as follows: we approximate $v(0;\lambda)$ and $v'(0;\lambda)$ using Theorem A.2; having these values we again apply Theorem A.2 to obtain a bound on $v(x;\lambda)$.

From (A.15), (A.16) and (A.23) we conclude that

$$v(0)=\begin{cases} K_2\sqrt{\omega'(b)}\left[\frac{3\xi(b)}{2r_1(0)}\right]^{1/6}\sin\beta\cos\left(\xi(b)+\frac{\pi}{12}\right)+O\left(\frac{e^{|\Im\xi(b)|}}{\lambda^{5/12}}\right), \\ \quad \sin\beta\neq 0, \\ \frac{K_2\cos\beta}{\lambda^{1/3}\sqrt{\omega'(b)}\left(\frac{3}{2}r_1(0)\xi(b)\right)^{1/6}}\sin\left(\xi(b)+\frac{\pi}{12}\right)+O\left(\frac{e^{|\Im\xi(b)|}}{\lambda^{11/12}}\right), \\ \quad \sin\beta=0, \end{cases}$$

and we get for $\sin\beta=0$

$$\begin{aligned} v'(0)=\frac{-\cos\beta}{\sqrt{\omega'(b)}\left(\frac{3}{2}\xi(b)\right)^{1/6}}\Bigg[&K_1\left(r_1(0)\right)^{1/6}\cos\left(\xi(b)-\frac{\pi}{12}\right) \\ &+\frac{K_2\omega''(0)}{2\lambda^{1/3}\sqrt{r_1(0)}}\sin\left(\xi(b)+\frac{\pi}{12}\right)+O\left(\frac{e^{|\Im\xi(b)|}}{\sqrt{\lambda}}\right)\Bigg], \end{aligned}$$

while for $\sin\beta\neq 0$ we obtain

$$\begin{aligned} v'(0)=\sin\beta\sqrt{\omega'(b)}\left(\frac{3}{2}\xi(b)\right)^{1/6}\Bigg[&K_1\lambda^{1/3}\left(r_1(0)\right)^{1/6}\sin\left(\xi(b)-\frac{\pi}{12}\right) \\ &-\frac{K_2\omega''(0)}{2\sqrt{r_1(0)}}\cos\left(\xi(b)+\frac{\pi}{12}\right)+O\left(\frac{e^{|\Im\xi(b)|}}{\lambda^{1/6}}\right)\Bigg]. \end{aligned}$$

The bounds for the Bessel functions $J_{1/3}$ and $J_{-1/3}$ given in [27, p. 49],

$$\left|J_s(z)\right|\leqslant K_s|z|^s e^{|\Im z|},\quad \forall z\in\mathbb{C}, \tag{3.15}$$

and [27, p. 579],

$$\left|J_s(z)\right|\leqslant K_s\frac{e^{|\Im z|}}{\sqrt{|z|}},\quad \forall z\in\mathbb{C}, \tag{3.16}$$

for some constant K_s, along with (A.15) and (A.17)–(A.22), enable us to conclude that

$$v(x)=O\left(\left[\left|\lambda^{1/3}v(0)\right|+\left|v'(0)\right|\right]\frac{e^{|\Im\xi|}}{\lambda^{1/3}}\right).$$

Hence we obtain

$$v(x)=\begin{cases} O\left(\frac{e^{|\Im\xi(x)|+|\Im\xi(b)|}}{\lambda^{5/12}}\right), & \sin\beta=0, \\ O\left(\lambda^{1/12}e^{|\Im\xi(x)|+|\Im\xi(b)|}\right), & \sin\beta\neq 0. \end{cases}$$

To establish bounds for $w(x)$ we distinguish two cases. Let M be a large constant, then $\sqrt{|x^3\lambda|}\geqslant M$ implies that $|\xi|$ is large and hence that the asymptotics of

(A.23) for y_1 and y_2 may be applied. (We note that $\sqrt{|x^3\lambda|}$ and ξ are of the same order of magnitude.)

Case IIa: $x \leqslant -|\lambda|^{-1/3}M$.

Using the asymptotic approximations from the Appendix of [14] we conclude that

$$w(x) = \begin{cases} O\left(\frac{\exp|\int_a^x \Im\sqrt{\lambda r(t)}\,dt|}{\sqrt{\lambda}r^{1/4}}\right), & \sin\alpha = 0, \\ O\left(\frac{\exp|\int_a^x \Im\sqrt{\lambda r(t)}\,dt|}{r^{1/4}}\right), & \sin\alpha \neq 0. \end{cases}$$

By the assumption that $x \leqslant -|\lambda|^{-1/3}M$ the above bounds yield

$$w(x) = \begin{cases} O\left(\frac{\exp|\int_a^x \Im\sqrt{\lambda r(t)}\,dt|}{\lambda^{5/12}}\right), & \sin\alpha = 0, \\ O\left(\lambda^{1/12}\exp\left|\int_a^x \Im\sqrt{\lambda r(t)}\,dt\right|\right), & \sin\alpha \neq 0. \end{cases}$$

Thus we obtain uniformly for $x \leqslant -|\lambda|^{-1/3}M$ and $\lambda \in \Gamma_n$ that

$$\frac{vw}{\psi} = O\left(\lambda^{-1/3}\right).$$

Case IIb: $-|\lambda|^{-1/3}M \leqslant x \leqslant 0$.

Proceeding as for v we obtain

$$w(0) = \begin{cases} O\left(\lambda^{-5/12}e^{|\Im\xi(a)|}\right), & \sin\alpha = 0, \\ O\left(\lambda^{1/12}e^{|\Im\xi(a)|}\right), & \sin\alpha \neq 0, \end{cases}$$

$$w'(0) = \begin{cases} O\left(\lambda^{-1/12}e^{|\Im\xi(a)|}\right), & \sin\alpha = 0, \\ O\left(\lambda^{5/12}e^{|\Im\xi(a)|}\right), & \sin\alpha \neq 0. \end{cases}$$

From (A.14), (A.16)–(A.18), (A.21) and the bound (3.15) we obtain

$$w(x) = \begin{cases} O\left(\lambda^{-5/12}e^{|\Im\xi(a)|+|\Im\xi(x)|}\right), & \sin\alpha = 0, \\ O\left(\lambda^{1/12}e^{|\Im\xi(a)|+|\Im\xi(x)|}\right), & \sin\alpha \neq 0. \end{cases}$$

Note that

$$\left|\Im\xi(a)\right| + \left|\Im\xi(x)\right| = \left|\Im\int_a^x \sqrt{\lambda r}\,dt\right| + 2\left|\Im\xi(x)\right|$$

and, as $-|\lambda|^{-1/3}M \leqslant x \leqslant 0$,

$$2\left|\Im\xi(x)\right| \leqslant c$$

for some constant c, independent of x and λ. Thus

$$O\left(e^{|\Im\xi(a)|+|\Im\xi(x)|}\right) = O\left(\exp\left|\Im\int_a^x \sqrt{\lambda r}\,dt\right|\right),$$

giving

$$w(x)=\begin{cases} O\big(\lambda^{-5/12}\exp\big|\Im\int_a^x\sqrt{\lambda r}\,dt\big|\big), & \sin\alpha=0,\\ O\big(\lambda^{1/12}\exp\big|\Im\int_a^x\sqrt{\lambda r}\,dt\big|\big), & \sin\alpha\neq 0,\end{cases}$$

and we obtain uniformly for $-|\lambda|^{-1/3}M\leqslant x\leqslant 0$ and $\lambda\in\Gamma_n$

$$\frac{vw}{\psi}=O\big(\lambda^{-1/3}\big).$$

This completes the proof of Theorem 3.1. □

4. A transformation operator

Throughout this section it will be assumed that $k_n=\tilde{k}_n$ and $\lambda_n=\tilde{\lambda}_n$ for all $n\in\Lambda$.

Let

$$H=D\ominus\{w_n\colon n\in\Lambda_0\},$$
$$\tilde{H}=D\ominus\{\tilde{w}_n\colon n\in\Lambda_0\}.$$

Define the transformation operator $T:H\to\tilde{H}$ by

$$Tw_n=\tilde{w}_n$$

for all $n\in\Lambda$, extended by linearity to the linear span of $\{w_n\colon n\in\Lambda\}$, which is a dense subset of H. (By the self-adjointness of $(A;\alpha;\beta)$ the w_n, $n\in\mathbb{Z}^0$, form a complete orthogonal basis for D.)

Proposition 4.1. *The operator $T:H\to\tilde{H}$ is bounded.*

Proof. Equations (A.13) and (A.28) give

$$|||\tilde{w}_n|||^2=\lambda_n\int_a^b r(\tilde{w}_n)^2\,dt=\lambda_n\mu(n,\alpha,\beta)\left[1+O\left(\frac{1}{n}\right)\right]$$

and

$$|||w_n|||^2=\lambda_n\int_a^b rw_n^2\,dt=\lambda_n\mu(n,\alpha,\beta)\left[1+O\left(\frac{1}{n}\right)\right].$$

Thus

$$\frac{|||Tw_n|||^2}{|||w_n|||^2}=1+O\left(\frac{1}{n}\right).\qquad\square$$

As

$$(\lambda R-A)w_n=(\lambda-\lambda_n)Rw_n,$$

we obtain

$$\frac{w_n}{\lambda - \lambda_n} = -S_\lambda w_n. \tag{4.1}$$

A similar relation is obviously valid for $\tilde{w}_n$.

Lemma 4.2. *On H,*

$$TS_\lambda = \tilde{S}_\lambda T$$

for $\lambda \neq \lambda_n, \tilde{\lambda}_n$, for all $n \in \mathbb{Z}^0$.

Proof. From (4.1), its analogue for $\tilde{l}$ and the definition of T it follows that

$$-TS_\lambda w_n = \frac{Tw_n}{\lambda - \lambda_n} = \frac{\tilde{w}_n}{\lambda - \lambda_n} = -\tilde{S}_\lambda \tilde{w}_n = -\tilde{S}_\lambda T w_n.$$

Hence the result follows since w_n, $n \in \Lambda$, form an orthonormal basis for H, and as S_λ and $\tilde{S}_\lambda$ are continuous and linear. □

The following corollary to Theorem 3.1 will be used at various points.

Corollary 4.3. *For all $n \in \Lambda$*

$$|||w_n|||^2 = \frac{\lambda_n \psi'(\lambda_n)}{k_n}.$$

Proof. Let $n \in \Lambda$ and λ not be an eigenvalue of $(A; \alpha; \beta)$. Then $-S_\lambda w_n = g_n$, where

$$g_n(x) = \frac{1}{\psi(\lambda)} \int_a^b G(x; y; \lambda) w_n(y) r(y)\, dy$$

and $G(x; y; \lambda)$ is as defined in (3.2). It is easily verified that g_n obeys the boundary conditions (1.2), (1.3) and that

$$Rw_n = (\lambda R - A) g_n. \tag{4.2}$$

From the definition of g_n and (3.3) of Theorem 3.1 one obtains

$$\begin{aligned} g_n(x) &= \sum_j \frac{w_j(x) k_j}{(\lambda - \lambda_j)\psi'(\lambda_j)} \int_a^b w_j w_n r\, dy \\ &= \sum_j \frac{w_j(x) k_j}{(\lambda - \lambda_j)\psi'(\lambda_j)\lambda_j} \int_a^b w_n l w_j\, dy \end{aligned}$$

$$= \sum_j \frac{w_j(x)k_j}{(\lambda - \lambda_j)\psi'(\lambda_j)\lambda_j} \langle w_j, w_n \rangle$$

$$= \frac{w_n(x)k_n}{(\lambda - \lambda_n)\psi'(\lambda_n)\lambda_n} |||w_n|||^2.$$

Thus by (4.1) we obtain

$$\frac{w_n(x)}{\lambda - \lambda_n} = g_n(x) = \frac{w_n(x)k_n}{(\lambda - \lambda_n)\psi'(\lambda_n)\lambda_n} |||w_n|||^2$$

from which the corollary follows if we note that the $w_n(x)$ are a.e. nonzero. □

At this point we can give an explicit expression for the operators in Lemma 4.2.

Lemma 4.4. *For $\lambda \neq \lambda_n$, for all $n \in \mathbb{Z}^0$, and $f \in H$*

$$-\tilde{S}_\lambda T f = h - \sum_{\Lambda_0} \frac{\tilde{u}_n \int_a^x w_n f r\,dt + \tilde{z}_n \int_x^b v_n f r\,dt}{(\lambda - \lambda_n)\psi'(\lambda_n)}, \tag{4.3}$$

where

$$\tilde{u}_n(x) = \tilde{v}(x; \lambda_n), \qquad \tilde{z}_n(x) = \tilde{w}(x; \lambda_n),$$

for $n \in \Lambda_0$, and, with K as defined in (3.2),

$$h(x; \lambda) = \frac{1}{\psi(\lambda)} \int_a^b K(x; y; \lambda) f(y) r(y)\,dy.$$

Proof. Let $\lambda \neq \lambda_n$, for all n, and $f \in H$. Then by (4.1), Theorem 2.4 and Corollary 4.3,

$$-TS_\lambda f = -\sum_\Lambda TS_\lambda w_n \frac{\langle f, w_n \rangle}{|||w_n|||^2} = \sum_\Lambda \frac{\langle f, w_n \rangle \tilde{w}_n}{(\lambda - \lambda_n)|||w_n|||^2}$$

$$= \sum_\Lambda \frac{\langle f, w_n \rangle \tilde{w}_n k_n}{(\lambda - \lambda_n)\psi'(\lambda_n)\lambda_n} = \sum_\Lambda \frac{\tilde{w}_n k_n}{(\lambda - \lambda_n)\psi'(\lambda_n)\lambda_n} \int_a^b f l w_n\,dt$$

$$= \sum_\Lambda \frac{\tilde{w}_n k_n}{(\lambda - \lambda_n)\psi'(\lambda_n)} \int_a^b r f w_n\,dt$$

$$= \sum_\Lambda \frac{k_n}{(\lambda - \lambda_n)\psi'(\lambda_n)} \left(\tilde{w}_n \int_a^x r f w_n\,dt + \tilde{w}_n \int_x^b r f w_n\,dt \right)$$

$$= \sum_\Lambda \frac{1}{(\lambda - \lambda_n)\psi'(\lambda_n)} \left(\tilde{v}_n \int_a^x r f w_n\,dt + \tilde{w}_n \int_x^b r f v_n\,dt \right).$$

But from (3.4) and the definition of h we have

$$h = \sum_{\Lambda_0} \frac{\tilde{u}_n \int_0^x w_n f r\, dy + \tilde{z}_n \int_x^1 v_n f r\, dy}{(\lambda - \lambda_n)\psi'(\lambda_n)} + \sum_{\Lambda} \frac{\tilde{v}_n \int_0^x w_n f r\, dy + \tilde{w}_n \int_x^1 v_n f r\, dy}{(\lambda - \lambda_n)\psi'(\lambda_n)},$$

which together with Lemma 4.2 concludes the proof. □

We are now ready for the main result of this section, which gives an expression for T itself.

Theorem 4.5. *For* $f \in H$

$$Tf(x) = f(x) - \sum_{\Lambda_0} \tilde{y}_n(x) \int_a^x w_n f r\, dy,$$

where

$$\tilde{y}_n = \frac{\tilde{u}_n - k_n \tilde{z}_n}{\psi'(\lambda_n)}$$

in the notation of Lemma 4.4.

Proof. From (3.5) of Theorem 3.1 and (2.7) we obtain

$$\frac{\tilde{v}w - \tilde{w}v}{\psi(\lambda)} = \sum_{\Lambda_0} \frac{\tilde{u}_n w_n - \tilde{z}_n v_n}{(\lambda - \lambda_n)\psi'(\lambda_n)}. \tag{4.4}$$

It follows from (4.4) and Lemma 4.4 that

$$-[\tilde{S}_\lambda Tf]' = \frac{\tilde{v}' \int_a^x w f r\, dy + \tilde{w}' \int_x^b v f r\, dy}{\psi(\lambda)} - \sum_{\Lambda_0} \frac{\tilde{u}_n' \int_a^x w_n f r\, dy + \tilde{z}_n' \int_x^b v_n f r\, dy}{(\lambda - \lambda_n)\psi'(\lambda_n)}. \tag{4.5}$$

Hence we obtain

$$\begin{aligned} RTf &= -(\lambda R - \tilde{A})\tilde{S}_\lambda Tf \\ &= r\frac{f(\tilde{v}'w - \tilde{w}'v)}{\psi(\lambda)} - r\sum_{\Lambda_0} \frac{f(\tilde{u}_n' w_n - \tilde{z}_n' v_n)}{(\lambda - \lambda_n)\psi'(\lambda_n)} \\ &\quad - r\sum_{\Lambda_0} \frac{\tilde{u}_n \int_a^x w_n f r\, dy + \tilde{z}_n \int_x^b v_n f r\, dy}{\psi'(\lambda_n)} \end{aligned}$$

giving, as r is a.e. nonzero,

$$Tf + \sum_{\Lambda_0} \frac{\tilde{u}_n \int_a^x w_n f r\,dy + \tilde{z}_n \int_x^b v_n f r\,dy}{\psi'(\lambda_n)}$$
$$= \frac{f(\tilde{v}'w - \tilde{w}'v)}{\psi(\lambda)} - \sum_{\Lambda_0} \frac{f(\tilde{u}_n' w_n - \tilde{z}_n' v_n)}{(\lambda - \lambda_n)\psi'(\lambda_n)}, \tag{4.6}$$

the left-hand side of which is independent of λ. For the remainder of the proof we assume that $\sin\alpha \neq 0 \neq \sin\beta$. The other cases can be handled in a similar manner.

We begin with the assumption that $r(0^-) < 0 < r(0^+)$. Let $\lambda = (\Theta_1(m))^2$, where $\Theta_1(m)$ is as in (3.10). From (A.1)–(A.4) and (3.12) it follows that

$$\begin{aligned}\tilde{v}'w - \tilde{w}'v &= [v'w - w'v] + \big[(\tilde{v}' - v')w - (\tilde{w}' - w')v\big]\\ &= \psi(\lambda) + \big[(\tilde{v}' - v')w - (\tilde{w}' - w')v\big]\\ &= \psi(\lambda)\big(1 + O\big(\lambda^{-1/2}\big)\big), \quad \text{for } \lambda \in \mathbb{R},\end{aligned}$$

whence

$$\frac{f(\tilde{v}'w - \tilde{w}'v)}{\psi(\lambda)} - \sum_{\Lambda_0} \frac{f(\tilde{u}_n' w_n - \tilde{z}_n' v_n)}{(\lambda - \lambda_n)\psi'(\lambda_n)} = \big(1 + O\big(\lambda^{-1/2}\big)\big) f. \tag{4.7}$$

Considering (4.6) and (4.7) we observe that the right-hand side of (4.7) is independent of λ, and is thus equal to f.

For $r(x) = xr_1(x)$, where $r_1(x) > 0$ for all $x \in [a, b]$, we merely replace Θ_1 of (3.10) by (3.13) and use and (A.14)–(A.23).

Hence for each case

$$Tf + \sum_{\Lambda_0} \frac{\tilde{u}_n \int_a^x w_n f r\,dy + \tilde{z}_n \int_x^b v_n f r\,dy}{\psi'(\lambda_n)} = f. \tag{4.8}$$

The proof is concluded by using (2.7), Theorem 2.4 and the orthogonality of v_n and f for $n \in \Lambda_0$. □

5. Main theorems

Our first main theorem generalizes one of Hochstadt [14] to (1.1) with indefinite r. We recall that $\mathbb{Z}^0 = \mathbb{Z}\backslash\{0\}$.

Theorem 5.1. *Let λ_n denote the eigenvalues of $(A;\alpha;\beta)$. Define $\tilde{\lambda}_n$ in a similar manner but with A replaced by $\tilde{A}$. Let γ satisfy $\sin(\beta - \gamma) \neq 0$, and suppose that $(A;\alpha;\gamma)$ and $(\tilde{A};\alpha;\gamma)$ have the same spectrum. Let $\Lambda_0 \subseteq \mathbb{Z}^0$ be a finite set and $\Lambda = \mathbb{Z}^0\backslash\Lambda_0$. If $\lambda_n = \tilde{\lambda}_n$, for all $n \in \Lambda$, then almost everywhere*

$$q = \tilde{q} + \sum_{\Lambda_0} \big[2r(\tilde{y}_n w_n)' + \tilde{y}_n w_n r'\big],$$

where $\tilde{y}_n$ *and* w_n *are suitable solutions of* $\tilde{l}\tilde{y} = \lambda_n r\tilde{y}$ *and* $ly = \lambda_n ry$*, respectively.*

Remark. If Λ_0 is empty, then we obtain an 'indefinite weight' version of Borg's classical result that two spectra determine q.

Proof. Let $n \in \Lambda$. Then $w_n \neq 0$ a.e. and from Lemma 4.2 we have

$$TS_\lambda w_n = \tilde{S}_\lambda T w_n,$$

for all $\lambda \neq \lambda_m, \tilde{\lambda}_m$, $m \in Z^0$, which with (4.1) and the definition of S_λ gives

$$(\lambda R - \tilde{A})Tw_n = RT(\lambda - \lambda_n)w_n.$$

Theorem 4.5 applied to the above yields

$$(\lambda R - \tilde{A})Tw_n = (\lambda - \lambda_n)rw_n - \sum_{m\in\Lambda_0} r\tilde{y}_m \int_a^x (\lambda - \lambda_n)rw_n w_m\,dt$$

which gives

$$(\lambda R - \tilde{A})Tw_n = (\lambda R - A)w_n - \sum_{m\in\Lambda_0} r\tilde{y}_m \int_a^x w_m(\lambda R - A)w_n\,dt.$$

Appealing once more to Theorem 4.5 we have

$$\tilde{A}\sum_{m\in\Lambda_0} \tilde{y}_m(x)\int_a^x w_m w_n r\,dt - \tilde{q}w_n = -qw_n + \sum_{m\in\Lambda_0} r\tilde{y}_m \int_a^x w_m A w_n\,dt.$$

Integrating by parts and using the boundary conditions at $x = a$ we obtain

$$\begin{aligned}&\tilde{A}\sum_{m\in\Lambda_0} \tilde{y}_m(x)\int_a^x w_m w_n r\,dt - \tilde{q}w_n \\ &\quad = -qw_n + \sum_{m\in\Lambda_0} r\tilde{y}_m\left[{w_m}'w_n - w_m w_n' + \int_a^x w_n A w_m\,dt\right].\end{aligned} \tag{5.1}$$

Noting that

$$\begin{aligned}\sum_{m\in\Lambda_0} r\tilde{y}_m\int_a^x w_n A w_m\,dt &= \sum_{m\in\Lambda_0} r\lambda_m\tilde{y}_m\int_a^x rw_n w_m\,dt \\ &= \sum_{m\in\Lambda_0}[\tilde{A}\tilde{y}_m]\int_a^x rw_n w_m\,dt\end{aligned}$$

from (5.1) we can deduce that

$$-\sum_{m\in\Lambda_0}\left[2\tilde{y}_m'(x)(w_m w_n r)+\tilde{y}_m(x)(w_m w_n r)'\right]-\tilde{q}w_n$$
$$=-qw_n+\sum_{m\in\Lambda_0} r\tilde{y}_m[{w_m}'w_n-w_m w_n'].$$

Thus

$$(q-\tilde{q})w_n=w_n\sum_{\Lambda_0}\left[2(\tilde{y}_m w_m)'r+\tilde{y}_m(x)w_m r'\right]. \qquad \square$$

From the material developed in Sections 2–4 we can in addition prove a Gelfand–Levitan type theorem. Again two scalar sequences determine q, this time a spectrum and corresponding eigenfunction norms.

Theorem 5.2. *Let λ_n and w_n denote the eigenvalues and eigenfunctions of $(A;\alpha;\beta)$, where w_n is normalised by $w_n(a)=\sin\alpha$, $w_n'(a)=-\cos\alpha$. Let $\rho_n=|||w_n|||$. Define $\tilde{\rho}_n$ and $\tilde{\lambda}_n$ in an analogous manner but with A replaced by $\tilde{A}$. Suppose that $\lambda_n=\tilde{\lambda}_n$ and $\rho_n=\tilde{\rho}_n$ for all $n\in\mathbb{Z}^0$. Then*

$$q=\tilde{q}.$$

Proof. From Lemma 2.3 applied to $(A;\alpha;\beta)$ and $(\tilde{A};\alpha;\beta)$ in place of $(A;\alpha;\gamma)$ and $(\tilde{A};\alpha;\gamma)$ we obtain $\psi=\tilde{\psi}$. Hence

$$\psi'(\lambda_n)=\tilde{\psi}'(\lambda_n) \tag{5.2}$$

for all $n\in Z^0$. Appealing to Theorem 4.3 with (5.2) and the assumption that $\rho_n=\tilde{\rho}_n$, we obtain $k_n=\tilde{k}_n$. The remainder of the proof is as for Theorem 5.1. $\square$

We now specialize to a case with certain symmetry properties. Without loss we take $[a,b]=[-1,1]$.

Theorem 5.3. *Let λ_n and w_n denote the eigenvalues and eigenfunctions of $(A;\alpha;\beta)$, where $\alpha+\beta=\pi$, $r(-x)=-r(x)$, $q(-x)=q(x)$ and w_n is normalised by $w_n(a)=\sin\alpha$, $w_n'(a)=-\cos\alpha$. Let $\rho_n=|||w_n|||$. Define $\tilde{\rho}_n$ and $\tilde{\lambda}_n$ in an analogous manner but with A replaced by $\tilde{A}$ (in which $\tilde{q}(-x)=\tilde{q}(x)$). Suppose that $\lambda_n=\tilde{\lambda}_n$ for $n\in Z^0$, with $n\geqslant 1$. Then*

$$q=\tilde{q}.$$

Proof. Straightforward calculation yields that $\lambda_{-n}=-\lambda_n$ and that $w_n(-x)$ is an eigenfunction for λ_{-n}. It thus follows that $v_n(x)=w_{-n}(-x)$ and hence

$$k_n w_n(x) = w_{-n}(-x), \tag{5.3}$$

$$k_{-n} w_{-n}(x) = w_n(-x). \tag{5.4}$$

Combining (5.3) and (5.4), $k_n k_{-n} w_n(x) = w_n(x)$ and thus

$$k_n k_{-n} = 1. \tag{5.5}$$

But $|||w_{-n}|||^2 = |||w_n|||^2$ and $|||w_{-n}|||^2 = |||v_n|||^2 = k_n^2 |||w_n|||^2$. So $k_n = \pm 1$ and by (5.5)

$$k_n = k_{-n} = \pm 1. \tag{5.6}$$

From Corollary 4.3

$$|||v_n|||\, |||w_n||| = |k_n|\, |||w_n|||^2 = \lambda_n \psi'(\lambda_n) \operatorname{sgn}(k_n),$$

thus

$$\operatorname{sgn}(k_n) = \operatorname{sgn}\bigl(\lambda_n \psi'(\lambda_n)\bigr). \tag{5.7}$$

However, if two problems of the type being considered have the same boundary conditions and the same eigenvalues then, as shown in Lemma 2.3, they have the same function ψ, i.e., $\tilde{\psi} = \psi$. Thus

$$k_n = \operatorname{sgn}(k_n) = \operatorname{sgn}\bigl(\lambda_n \psi'(\lambda_n)\bigr) = \operatorname{sgn}\bigl(\tilde{\lambda}_n \tilde{\psi}'(\tilde{\lambda}_n)\bigr) = \operatorname{sgn}(\tilde{k}_n) = \tilde{k}_n.$$

The remainder of the proof is as for Theorem 5.1. □

Remark. Theorem 5.3 thus enables us to conclude that for this semi-symmetric case we need only the positive part of one spectrum to determine the potential uniquely.

Appendix A

It should be noted that when we refer to $[e^{i\theta}]^s$ for $\theta \in (-\pi, \pi]$ and $s > 0$ we mean the branch $e^{is\theta}$. In particular, we use the notation $\sqrt{\lambda r(t)}$ to mean the complex function with argument in $(-\pi/2, \pi/2]$.

Theorem A.1. *For the case of* $r(0^-) < 0 < r(0^+)$. *Let*

$$\bar{r}(x) = \begin{cases} r(0^+), & x > 0, \\ r(0^-), & x < 0, \end{cases} \qquad \rho(x) = \left(\frac{r(x)}{\bar{r}(x)}\right)^{1/4},$$

$E(x, y) = \exp|\Im[\xi(x) - \xi(y)]|$ *and* $\xi(x) = \int_0^x \sqrt{\lambda r(\tau)}\, d\tau$, *where* $\lambda \in \mathbb{C}$. *Finally, let*

$$\kappa = \arctan \sqrt{\left|\frac{r(0^-)}{r(0^+)}\right|} \in \left(0, \frac{\pi}{2}\right).$$

Then as $|\lambda| \to \infty$, *for* $\sin\alpha \neq 0$ *we obtain*

$$w(x) = \frac{\rho(a)}{\rho(x)} \sin\alpha \left[\cos\xi(x)\cos\xi(a) + \sqrt{\frac{\bar r(a)}{\bar r(x)}} \sin\xi(x)\sin\xi(a) + O\left(\frac{E(x,a)}{\sqrt{\lambda}}\right)\right], \tag{A.1}$$

$$w'(x) = -\rho(a)\rho(x)\sin\alpha\sqrt{\lambda\bar r(x)} \left[\sin\xi(x)\cos\xi(a) - \sqrt{\frac{\bar r(a)}{\bar r(x)}} \cos\xi(x)\sin\xi(a) + O\left(\frac{E(x,a)}{\sqrt{\lambda}}\right)\right], \tag{A.2}$$

and for $\sin\alpha = 0$

$$w(x) = \frac{\cos\alpha}{\sqrt{\lambda\bar r(a)}\rho(a)\rho(x)} \left[\cos\xi(x)\sin\xi(a) - \sqrt{\frac{\bar r(a)}{\bar r(x)}} \sin\xi(x)\cos\xi(a) + O\left(\frac{E(x,a)}{\sqrt{\lambda}}\right)\right], \tag{A.3}$$

$$w'(x) = -\frac{\rho(x)}{\rho(a)} \cos\alpha \left[\cos\xi(x)\cos\xi(a) + \sqrt{\frac{\bar r(x)}{\bar r(a)}} \sin\xi(x)\sin\xi(a) + O\left(\frac{E(x,a)}{\sqrt{\lambda}}\right)\right], \tag{A.4}$$

for $\sin\beta \neq 0$

$$v(x) = \frac{\rho(b)}{\rho(x)} \sin\beta \left[\cos\xi(x)\cos\xi(b) + \sqrt{\frac{\bar r(b)}{\bar r(x)}} \sin\xi(x)\sin\xi(b) + O\left(\frac{E(b,x)}{\sqrt{\lambda}}\right)\right], \tag{A.5}$$

$$v'(x) = -\rho(b)\rho(x)\sin\beta\sqrt{\lambda\bar r(x)} \left[\cos\xi(b)\sin\xi(x) - \sqrt{\frac{\bar r(b)}{\bar r(x)}} \cos\xi(x)\sin\xi(b) + O\left(\frac{E(b,x)}{\sqrt{\lambda}}\right)\right], \tag{A.6}$$

and for $\sin\beta = 0$

$$
\begin{aligned}
v(x) = {} & \frac{\cos\beta}{\sqrt{\lambda\bar{r}(b)}\rho(b)\rho(x)}\Bigg[\cos\xi(x)\sin\xi(b) \\
& - \sqrt{\frac{\bar{r}(b)}{\bar{r}(x)}}\sin\xi(x)\cos\xi(b) + O\left(\frac{E(b,x)}{\sqrt{\lambda}}\right)\Bigg],
\end{aligned}
\tag{A.7}
$$

$$
\begin{aligned}
v'(x) = {} & -\frac{\rho(x)}{\rho(b)}\cos\beta\Bigg[\cos\xi(x)\cos\xi(b) \\
& + \sqrt{\frac{\bar{r}(x)}{\bar{r}(b)}}\sin\xi(x)\sin\xi(b) + O\left(\frac{E(b,x)}{\sqrt{\lambda}}\right)\Bigg],
\end{aligned}
\tag{A.8}
$$

where the constants implied by the symbol $O(\cdot)$ *are independent of* x; *for* $\sin\alpha \neq 0$

$$
\psi(\lambda) = \begin{cases}
-\rho(a)\rho(b)\sin\alpha\sin\beta\big[\sqrt{\lambda\bar{r}(b)}\cos\xi(a)\sin\xi(b) \\
\quad - \sqrt{\lambda\bar{r}(a)}\sin\xi(a)\cos\xi(b)\big] + O(E(b,a)), \quad 0 \neq \sin\beta, \\
\frac{\rho(a)}{\rho(b)}\sin\alpha\cos\beta\big[\cos\xi(a)\cos\xi(b) + \sqrt{\frac{\lambda\bar{r}(a)}{\lambda\bar{r}(b)}}\sin\xi(a)\sin\xi(b)\big] \\
\quad + O\big(\frac{E(b,a)}{\sqrt{\lambda}}\big), \quad 0 = \sin\beta,
\end{cases}
\tag{A.9}
$$

while for $\sin\alpha = 0$

$$
\psi(\lambda) = \begin{cases}
\frac{\cos\alpha\cos\beta}{\rho(a)\rho(b)}\big[\frac{\sin\xi(a)\cos\xi(b)}{\sqrt{\lambda\bar{r}(a)}} - \frac{\cos\xi(a)\sin\xi(b)}{\sqrt{\lambda\bar{r}(b)}}\big] \\
\quad + O\big(\frac{E(b,a)}{\lambda}\big), \quad 0 = \sin\beta, \\
-\frac{\rho(b)}{\rho(a)}\cos\alpha\sin\beta\big[\sqrt{\frac{\lambda\bar{r}(b)}{\lambda\bar{r}(a)}}\sin\xi(a)\sin\xi(b) + \cos\xi(a)\cos\xi(b)\big] \\
\quad + O\big(\frac{E(b,a)}{\sqrt{\lambda}}\big), \quad 0 \neq \sin\beta.
\end{cases}
\tag{A.10}
$$

For $n \to \infty$

$$
\sqrt{\lambda_n} = \begin{cases}
\frac{\kappa+(n-1)\pi}{\int_0^b \sqrt{r(t)}\,dt} + O\big(\frac{1}{n}\big), & \sin\beta \neq 0, \\
\frac{\kappa+(n-1/2)\pi}{\int_0^b \sqrt{r(t)}\,dt} + O\big(\frac{1}{n}\big), & \sin\beta = 0,
\end{cases}
\tag{A.11}
$$

$$
\sqrt{-\lambda_{-n}} = \begin{cases}
\frac{-\kappa+(n-1/2)\pi}{\int_a^0 \sqrt{-r(t)}\,dt} + O\big(\frac{1}{n}\big), & \sin\alpha \neq 0, \\
\frac{-\kappa+n\pi}{\int_a^0 \sqrt{-r(t)}\,dt} + O\big(\frac{1}{n}\big), & \sin\alpha = 0,
\end{cases}
\tag{A.12}
$$

and for $|n| \to \infty$

$$
\int_a^b r w_n^2 = \mu(n,\alpha,\beta)\left[1 + O\left(\frac{1}{n}\right)\right],
\tag{A.13}
$$

where $\operatorname{sgn}(n)\mu(n,\alpha,\beta)$ *is positive and independent of* q.

Proof. The results for $r(0^-) < 0 < r(0^+)$ follow directly from the asymptotics in the Appendix of [14] applied on the intervals $[a, 0]$ and $[0, b]$ with our equation written respectively in the form

$$-u'' + qu = \bigl(r(0^-)\lambda\bigr)\frac{r(x)}{r(0^-)}u$$

and

$$-u'' + qu = \bigl(r(0^+)\lambda\bigr)\frac{r(x)}{r(0^+)}u. \qquad \square$$

Theorem A.2. *For the case of $r(x) = xr_1(x)$, where $r_1(x) > 0$, $\forall x \in [a, b]$, let $K_1 = \Gamma(2/3)3^{1/6}/\sqrt{\pi}$, $K_2 = \Gamma(1/3)/(3^{1/6}\sqrt{\pi}\,)$,*

$$\omega(x) = \left(\frac{3}{2}\int_0^x \sqrt{r(t)}\,dt\right)^{2/3} \quad \textit{and} \quad \xi(x) = \int_0^x \sqrt{\lambda r(\tau)}\,d\tau.$$

Then

$$\begin{bmatrix} w \\ w' \end{bmatrix} = \lambda^{-1/3}\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} y_2'(a) & -y_2(a) \\ -y_1'(a) & y_1(a) \end{bmatrix}\begin{bmatrix} \sin\alpha \\ -\cos\alpha \end{bmatrix}, \tag{A.14}$$

$$\begin{bmatrix} v \\ v' \end{bmatrix} = \lambda^{-1/3}\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} y_2'(b) & -y_2(b) \\ -y_1'(b) & y_1(b) \end{bmatrix}\begin{bmatrix} \sin\beta \\ -\cos\beta \end{bmatrix}, \tag{A.15}$$

where y_1 and y_2 obey the initial conditions:

$$\begin{bmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{bmatrix} = \begin{bmatrix} [r_1(0)]^{-1/6} & 0 \\ -\frac{\omega''(0)}{2\sqrt{r_1(0)}} & \lambda^{1/3}[r_1(0)]^{1/6} \end{bmatrix}. \tag{A.16}$$

For $|\lambda| \to \infty$ we have

$$y_1 = \frac{\Gamma(2/3)}{3^{1/3}\sqrt{\omega'}}\left(\frac{3}{2}\xi\right)^{1/3} J_{-1/3}(\xi) + O\bigl(E(x;\lambda)\bigr), \tag{A.17}$$

$$y_2 = \frac{3^{1/3}\Gamma(4/3)}{\sqrt{\omega'}}\left(\frac{3}{2}\xi\right)^{1/3} J_{1/3}(\xi) + O\bigl(E(x;\lambda)\bigr), \tag{A.18}$$

$$\begin{aligned} y_1' = {} & -\frac{\lambda^{1/3}\sqrt{\omega'}\,\Gamma(2/3)}{3^{1/3}}\left(\frac{3}{2}\xi\right)^{2/3} J_{2/3}(\xi) \\ & - \frac{\omega''\Gamma(2/3)}{2[\omega']^{3/2}3^{1/3}}\left(\frac{3}{2}\xi\right)^{1/3} J_{-1/3}(\xi) + O\bigl(E'(x;\lambda)\bigr), \end{aligned} \tag{A.19}$$

$$\begin{aligned} y_2' = {} & 3^{1/3}\lambda^{1/3}\sqrt{\omega'}\,\Gamma(4/3)\left(\frac{3}{2}\xi\right)^{2/3} J_{-2/3}(\xi) \\ & - \frac{3^{1/3}\omega''\Gamma(4/3)}{2[\omega']^{3/2}}\left(\frac{3}{2}\xi\right)^{1/3} J_{1/3}(\xi) + O\bigl(E'(x;\lambda)\bigr), \end{aligned} \tag{A.20}$$

where

$$E(x;\lambda)=\frac{\sqrt{x}\,\xi^{1/3}[|J_{-1/3}(\xi)|+|J_{1/3}(\xi)|]}{\sqrt{\lambda}},\tag{A.21}$$

$$E'(x;\lambda)=\frac{\sqrt{x}\,\xi^{2/3}[|J_{-2/3}(\xi)|+|J_{2/3}(\xi)|]}{\lambda^{1/6}};\tag{A.22}$$

in particular for $|\xi|\to\infty$

$$\begin{aligned}\begin{bmatrix} y_1 & y_2\\ y_1' & y_2'\end{bmatrix}&=\begin{bmatrix}\frac{1}{\sqrt{\omega'}\,[\frac{3}{2}\xi]^{1/6}} & 0\\ 0 & \lambda^{1/3}\sqrt{\omega'}\,\big[\frac{3}{2}\xi\big]^{1/6}\end{bmatrix}\\ &\quad\times\left(\begin{bmatrix}K_1\cos\big(\xi-\frac{\pi}{12}\big) & K_2\sin\big(\xi+\frac{\pi}{12}\big)\\ -K_1\sin\big(\xi-\frac{\pi}{12}\big) & K_2\cos\big(\xi+\frac{\pi}{12}\big)\end{bmatrix}+O\left(\frac{e^{|\Im\xi|}}{\xi}\right)\right).\end{aligned}\tag{A.23}$$

For $|\lambda|\to\infty$*, if* $0\neq\sin\beta$*, then*

$$\psi(\lambda)=\begin{cases}\lambda^{1/3}K_1K_2\sin\alpha\sin\beta\sqrt{\omega'(a)\omega'(b)}\,\big(\frac{9}{4}\xi(a)\xi(b)\big)^{1/6}\\ \quad\times\Big[\cos\big(\xi(a)+\frac{\pi}{12}\big)\sin\big(\xi(b)-\frac{\pi}{12}\big)\\ \quad\quad-\cos\big(\xi(b)+\frac{\pi}{12}\big)\sin\big(\xi(a)-\frac{\pi}{12}\big)\\ \quad\quad+O\big(\frac{e^{|\Im\xi(a)|+|\Im\xi(b)|}}{\sqrt{\lambda}}\big)\Big], \quad \sin\alpha\neq 0,\\ K_1K_2\cos\alpha\sin\beta\sqrt{\frac{\omega'(b)}{\omega'(a)}}\,\big(\frac{\xi(b)}{\xi(a)}\big)^{1/6}\\ \quad\times\Big[\sin\big(\xi(a)+\frac{\pi}{12}\big)\sin\big(\xi(b)-\frac{\pi}{12}\big)\\ \quad\quad-\cos\big(\xi(b)+\frac{\pi}{12}\big)\cos\big(\xi(a)-\frac{\pi}{12}\big)\\ \quad\quad+O\big(\frac{e^{|\Im\xi(a)|+|\Im\xi(b)|}}{\sqrt{\lambda}}\big)\Big], \quad \sin\alpha=0,\end{cases}\tag{A.24}$$

while for $0=\sin\beta$

$$\psi(\lambda)=\begin{cases}\frac{K_1K_2\cos\alpha\cos\beta}{\lambda^{1/3}\sqrt{\omega'(a)\omega'(b)}\,(\frac{9}{4}\xi(a)\xi(b))^{1/6}}\\ \quad\times\Big[\sin\big(\xi(a)+\frac{\pi}{12}\big)\cos\big(\xi(b)-\frac{\pi}{12}\big)\\ \quad\quad-\sin\big(\xi(b)+\frac{\pi}{12}\big)\cos\big(\xi(a)-\frac{\pi}{12}\big)\\ \quad\quad+O\big(\frac{e^{|\Im\xi(a)|+|\Im\xi(b)|}}{\sqrt{\lambda}}\big)\Big], \quad \sin\alpha=0,\\ K_1K_2\sin\alpha\cos\beta\sqrt{\frac{\omega'(a)}{\omega'(b)}}\,\big(\frac{\xi(a)}{\xi(b)}\big)^{1/6}\\ \quad\times\Big[\cos\big(\xi(a)+\frac{\pi}{12}\big)\cos\big(\xi(b)-\frac{\pi}{12}\big)\\ \quad\quad-\sin\big(\xi(b)+\frac{\pi}{12}\big)\sin\big(\xi(a)-\frac{\pi}{12}\big)\\ \quad\quad+O\big(\frac{e^{|\Im\xi(a)|+|\Im\xi(b)|}}{\sqrt{\lambda}}\big)\Big], \quad \sin\alpha\neq 0.\end{cases}\tag{A.25}$$

For $n \to \infty$

$$\sqrt{\lambda_n} = \begin{cases} \frac{\pi(n-1/4)}{\int_0^b \sqrt{r}} + O\left(\frac{1}{n}\right), & \sin\beta \neq 0, \\ \frac{\pi(n-3/4)}{\int_0^b \sqrt{r}} + O\left(\frac{1}{n}\right), & \sin\beta = 0, \end{cases} \tag{A.26}$$

$$\sqrt{-\lambda_{-n}} = \begin{cases} -\frac{\pi(n-1/4)}{\int_a^0 \sqrt{-r}} + O\left(\frac{1}{n}\right), & \sin\alpha \neq 0, \\ -\frac{\pi(n-3/4)}{\int_a^0 \sqrt{-r}} + O\left(\frac{1}{n}\right), & \sin\alpha = 0, \end{cases} \tag{A.27}$$

and for $|n| \to \infty$

$$\int_a^b r w_n^2 = \mu(n, \alpha, \beta)\left[1 + O\left(\frac{1}{n}\right)\right], \tag{A.28}$$

where $\operatorname{sgn}(n)\mu(n, \alpha, \beta)$ *is positive and independent of* q.

Proof. For the case of $r(x) = x r_1(x)$, where $r_1(x) > 0$, $\forall x \in [a, b]$, the results are from [11]. The stated result for ψ can be calculated by noting that

$$\psi(\lambda) = \det\begin{bmatrix} v(0) & w(0) \\ v'(0) & w'(0) \end{bmatrix}.$$

The eigenvalue asymptotics come from [11, Eqs. (2.42) and (2.43)] while the result concerning the norms of the eigenfunctions is a weaker form of [11, Eq. (3.25)]. □

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